- **矢量分析**

场的分类：数量场$u(x,y,z)$,矢量场$\vec{u}(x,y,z)=P(x,y,z)\vec{i}+Q(x,y,z)\vec{j}+R(x,y,z)\vec{k}$

后请注意区分

Nabla 算符 ∇　若记为向量$\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$,则各运算法则与矢量运算相同

方向导数：数量场$u(x,y,z)$沿某一方向$\vec{l}=(\cos\alpha,\cos\beta,\cos\gamma)$的增减情况和速度

$$\frac{\partial u}{\partial\vec{l}}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma$$

梯度：数量场→矢量场，方向表示该点方向导数最大的方向（增加最快的方向），大小表示增加最快方向导数的值（增加的速度）

$$\operatorname{grad}u=\nabla u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$$

注：$\frac{\partial u}{\partial\vec{l}}=\nabla u\cdot\vec{l}=|\nabla u||\vec{l}|\cos\theta,|\vec{l}|=1,\cos\theta\in[-1,1]$

$\therefore\nabla u$与l方向相同时方向导数有最大值$\left(\frac{\partial u}{\partial\vec{l}}\right)_M=|\nabla u|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2}$

方向相反时有负的最大值,垂直时$\frac{\partial u}{\partial\vec{l}}=0$

散度：矢量场→数量场，定义为通量与体积比的极限（定义了解即可）

$\operatorname{div}\vec{u}=\nabla\cdot\vec{u}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z},\operatorname{div}\vec{u}\equiv 0$ 的场称为无源场

旋度：矢量场→矢量场

方向旋量：取定一点及一个方向$\vec{n}=(\cos\alpha,\cos\beta,\cos\gamma)$，在过该点的垂面上作闭路环绕该点，环量与面积比的极限称为该点绕该方向的方向旋量。（定义了解即可）

设$\vec{u}=\big(P(x,y,z),Q(x,y,z),R(x,y,z)\big)$

$$R_n=\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\vec{n}\cdot(\nabla\times\vec{u})$$

旋度的方向定义为使方向旋量达到最大的方向，大小为此时的方向旋量

$\operatorname{rot}\vec{u}=\nabla\times\vec{u}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$若$\operatorname{rot}\vec{u}\equiv 0$,称$\vec{u}$为无旋场

注：$R_n=\vec{n}\cdot\operatorname{rot}\vec{u}$,故$\vec{n}$与$\operatorname{rot}\vec{u}$ 方向相同时R_n有最大值$|\operatorname{rot}\vec{u}|$

保守场：沿任一闭路的环量为零。

在单连通区域内，保守场与无旋场等价。

- **例** *1*

$u = 2x + 3y + 4z$

$\operatorname{grad} u = \nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = (2,3,4) = 2\vec{i} + 3\vec{j} + 4\vec{k}$

$\operatorname{div}\operatorname{grad} u = \nabla \cdot (\nabla u) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$, 其中$P = \frac{\partial u}{\partial x} = 2$，$Q = \frac{\partial u}{\partial y} = 3$，$R = \frac{\partial u}{\partial z} = 4$

于是$\operatorname{div}\operatorname{grad} u = 0 + 0 + 0 = 0$

$\operatorname{div}\operatorname{grad} u = \nabla \cdot (\nabla u)$, 记为 $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial z^2}$

$$\operatorname{rot}\operatorname{grad} u = \nabla \times (\nabla u) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix} = 0$$

此式对任意数量场均成立，即梯度场是无旋场

- **例** *2*

$\vec{u} = (x + y)\vec{i} + (y + z)\vec{j} + (z + x)\vec{k}$

$\operatorname{div}\vec{u} = \nabla \cdot \vec{u} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 1 + 1 + 1 = 3$

$$\operatorname{rot}\vec{u} = \nabla \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (-1, -1, -1)$$

$$\operatorname{div}\operatorname{rot}\vec{u} = \nabla \cdot (\nabla \times \vec{u}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0$$

此式对任意矢量场均适用，即旋度场是无源场。

梯度，散度，旋度的复合运算，可依照矢量运算和求导法则推导

以下函数f为从数量到数量的函数, u, v为数量场函数, $\vec{u}, \vec{v}$为矢量场函数, a, b为常数

线性性质：

$a1)\nabla(au+bv)=a\nabla u+b\nabla v$

$b1)\nabla\cdot(a\vec{u}+b\vec{v})=a\nabla\cdot\vec{u}+b\nabla\cdot\vec{v}$

$c1)\nabla\times(a\vec{u}+b\vec{v})=a\nabla\times\vec{u}+b\nabla\times\vec{v}$

数量函数乘法性质：

$a2)\nabla(uv)=v(\nabla u)+u(\nabla v)$

$b2)\nabla\cdot(u\vec{v})=u\nabla\cdot\vec{v}+\vec{v}\cdot\nabla u$

$c2)\nabla\times(u\vec{v})=u(\nabla\times\vec{v})+(\nabla u)\times\vec{v}$

注：可直接按照乘法求导的微分法记，注意验证最后的结果是矢量还是数量。

复合性质：

$a3)\nabla\big(f(u)\big)=f'(u)\nabla u$

矢量函数乘法性质：

$b3)\nabla\cdot(\vec{u}\times\vec{v})=\vec{v}\cdot(\nabla\times\vec{u})-\vec{u}\cdot(\nabla\times\vec{v})$

$c3)\nabla\times(\vec{u}\times\vec{v})=(\nabla\cdot\vec{v})\vec{u}-(\nabla\cdot\vec{u})\vec{v}$

推导示范：

$b2)\nabla\cdot(u\vec{v})=u\nabla\cdot\vec{v}+\vec{v}\cdot\nabla u$

证：设$\vec{v}=\left(v_x,v_y,v_z\right)$，其中$v_x,v_y,v_z$均是$x,y,z$的数量函数

$$\nabla\cdot(u\vec{v})=\frac{\partial uv_x}{\partial x}+\frac{\partial uv_x}{\partial y}+\frac{\partial uv_z}{\partial z}=\left(\frac{\partial u}{\partial x}v_x+\frac{\partial v_x}{\partial x}u\right)+\left(\frac{\partial u}{\partial y}v_y+\frac{\partial v_y}{\partial y}u\right)+\left(\frac{\partial u}{\partial z}v_z+\frac{\partial v_z}{\partial z}u\right)$$
$$=\left(\frac{\partial v_x}{\partial x}u+\frac{\partial v_y}{\partial y}u+\frac{\partial v_z}{\partial z}u\right)+\left(\frac{\partial u}{\partial x}v_x+\frac{\partial u}{\partial y}v_y+\frac{\partial u}{\partial z}v_z\right)$$
$$=u\nabla\cdot\vec{v}+\vec{v}\cdot(\nabla u)$$

$c3)\nabla\times(\vec{u}\times\vec{v})=(\nabla\cdot\vec{v})\vec{u}-(\nabla\cdot\vec{u})\vec{v}$

证：需要用到公式$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$，此式证略（其实是不会）

代入即可得到$\nabla\times(\vec{u}\times\vec{v})=(\nabla\cdot\vec{v})\vec{u}-(\nabla\cdot\vec{u})\vec{v}$

- **例 3**

$\vec{c}$为常向量,$\vec{r}=x\vec{\imath}+y\vec{\jmath}+z\vec{k},r=|\vec{r}|$

$\operatorname{div}(\vec{c}\times f(r)\vec{r})=\nabla\cdot\left(\vec{c}\times(f(r)\vec{r})\right)=(f(r)\vec{r})\cdot(\nabla\times\vec{c})-\vec{c}\cdot\left(\nabla\times(f(r)\vec{r})\right)$

$=(f(r)\vec{r})\cdot\vec{0}-\vec{c}\cdot\left(f(r)(\nabla\times\vec{r})+\vec{r}\times\left(\nabla f(r)\right)\right)$

$\nabla\times\vec{r}=\nabla\times(x,y,z)=\vec{0}$

$\nabla f(r)=f'(r)\nabla(r)$

$\nabla r=\nabla\left(\sqrt{x^2+y^2+z^2}\right)=\left(\frac{x}{r},\frac{y}{r},\frac{z}{r}\right)=\frac{\vec{r}}{r}$

原式$=\vec{c}\cdot\left(\vec{r}\times\left(f'(r)\frac{\vec{r}}{r}\right)\right)=0\quad[\because\vec{r}\times\vec{r}=\vec{0}]$

注：一些重要结论

$\nabla r=\left(\frac{x}{r},\frac{y}{r},\frac{z}{r}\right)=\frac{\vec{r}}{r}$

$\nabla\times\vec{r}=\nabla\times(x,y,z)=\vec{0}$

$\nabla\times\vec{c}=\vec{0}$

- **各类积分**

- **第一型曲线积分**

$L:\left(x(t),y(t),z(t)\right),a\le t\le b$

$\int_L f(x,y,z)\,\mathrm{d}s=\int_a^b f[x(t),y(t),z(t)]\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\mathrm{d}t$

即 $\mathrm{d}s=\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\mathrm{d}t\quad\left[\because\mathrm{d}s=\sqrt{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2}\right]$

- **第二型曲线积分**

$\vec{f}=P(x,y,z)\vec{\imath}+Q(x,y,z)\vec{\jmath}+R(x,y,z)\vec{k}$

$\int_{\widehat{AB}}P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z=\int_a^b[Px'(t)+Qy'(t)+Rz'(t)]\mathrm{d}t$

即 $\mathrm{d}x=x'(t)\mathrm{d}t,\mathrm{d}y=y'(t)\mathrm{d}t,\mathrm{d}z=z'(t)\mathrm{d}t$

- **第一型曲面积分**

$S:\left(x(u,v),y(u,v),z(u,v)\right),(u,v)\in\Omega$

$\overrightarrow{r_u'} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$ $\overrightarrow{r_v'} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$

$A = \frac{\partial(y,z)}{\partial(u,v)}, B = \frac{\partial(z,x)}{\partial(u,v)}, C = \frac{\partial(x,y)}{\partial(u,v)}$

$E = \overrightarrow{r_u'} \cdot \overrightarrow{r_u'}, F = \overrightarrow{r_u'} \cdot \overrightarrow{r_v'}, G = \overrightarrow{r_v'} \cdot \overrightarrow{r_v'}$

$\iint_S f(x,y,z)\mathrm{d}S = \iint_\Omega f\big(x(u,v), y(u,v), z(u,v)\big)\sqrt{EG - F^2}\,\mathrm{d}u\mathrm{d}v$

注:$\sqrt{EG-F^2}$其实就是$\left|\overrightarrow{r_u'} \times \overrightarrow{r_v'}\right| = \left|\overrightarrow{r_u'}\right|\left|\overrightarrow{r_v'}\right|\sin\theta$

- **第二型曲面积分**

$\vec{f} = P(x,y,z)\vec{\imath} + Q(x,y,z)\vec{\jmath} + R(x,y,z)\vec{k}$

$\iint_S P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \pm\iint_\Omega (PA + QB + RC)\mathrm{d}u\mathrm{d}v$

正负号由取定曲面的侧决定。

注: $\iint_S P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y$ 可拆成三个积分之和

$\iint_S P\mathrm{d}y\mathrm{d}z$ 可化为在S的yOz面上投影区域上的积分, 并用y,z表示x, 即

$\iint_S P\mathrm{d}y\mathrm{d}z = \pm\iint_{S_x} P(x(y,z),y,z)\mathrm{d}y\mathrm{d}z$

- ***例 1***

计算$\int_L (xy + yz + zx)\,\mathrm{d}s$, L为球面$x^2+y^2+z^2=a^2$与平面$x+y+z=0$ 的交线

联立可得 $2x^2 + 2xy + 2y^2 = a^2 \Rightarrow \left(\sqrt{2}x + \frac{y}{\sqrt{2}}\right)^2 + \left(\sqrt{\frac{3}{2}}y\right)^2 = a^2$

设$\sqrt{2}x + \frac{y}{\sqrt{2}} = a\cos\theta$, $\sqrt{\frac{3}{2}}y = a\sin\theta$

$$\begin{cases} x = \frac{a\left(\sqrt{3}\cos\theta - \sin\theta\right)}{\sqrt{6}} \\ y = \frac{a\sin\theta\sqrt{2}}{\sqrt{3}} \\ z = -\frac{a\left(\sqrt{3}\cos\theta + \sin\theta\right)}{\sqrt{6}} \end{cases}, \begin{cases} x' = \frac{a\left(-\sqrt{3}\sin\theta - \cos\theta\right)}{\sqrt{6}} \\ y' = \frac{a\cos\theta\sqrt{2}}{\sqrt{3}} \\ z' = -\frac{a\left(-\sqrt{3}\sin\theta + \cos\theta\right)}{\sqrt{6}} \end{cases}$$

原式 $= \int_0^{2\pi} -\frac{a^2}{2}\sqrt{a^2}\mathrm{d}\theta = -\pi a^3$

➢ **例 2**

计算$\oint(x^2+y^2)\mathrm{d}x+(x+y)^2\mathrm{d}y$,积分路径$L$为$x^2+y^2=ax$,逆时针方向

极坐标换元,$r^2=ar\cos\theta\Rightarrow r=a\cos\theta\Rightarrow\begin{cases}x=a\cos^2\theta\\y=a\cos\theta\sin\theta\end{cases}$

原式$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(-2a^3\cos^3\theta\sin\theta+a^3\cos^2\theta\,(1+2\sin\theta\cos\theta)(\cos^2\theta-\sin^2\theta)\right)\mathrm{d}\theta$

$=a^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(-2\cos^3\theta\sin\theta+\cos^4\theta+2\cos^5\theta\sin\theta-\cos^2\theta\sin^2\theta-2\cos^3\theta\sin^3\theta)\mathrm{d}\theta$

$=\frac{1}{4}\pi a^3$

➢ **例 3**

计算$\iint_D|xyz|\mathrm{d}S$,S为曲面$z=x^2+y^2$被平面$z=1$ 截下的部分

参数化表示$\begin{cases}x=u\\y=v\\z=u^2+v^2\end{cases}$,$\sqrt{EG-F^2}=\sqrt{4u^2+4v^2+1}$

原式$=4\iint_D(u^3v+uv^3)\sqrt{4u^2+4v^2+1}\mathrm{d}u\mathrm{d}v$,可知投影$D$为四分之一圆

再用极坐标换元得原式$=4\times\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\,\mathrm{d}\theta\int_0^1 r^5\sqrt{4r^2+1}\mathrm{d}r=\frac{125\sqrt{5}-1}{420}$

➢ **例 4**

计算$\iint_S x^3\mathrm{d}y\mathrm{d}z+y^3\mathrm{d}z\mathrm{d}x+z^3\mathrm{d}x\mathrm{d}y$,$S$为球面$x^2+y^2+z^2=R^2$的外侧

先算$\iint_S z^3\mathrm{d}x\mathrm{d}y=2\iint_D z^3\mathrm{d}x\mathrm{d}y=2\iint_D(R^2-x^2-y^2)^{\frac{3}{2}}\mathrm{d}x\mathrm{d}y$,投影$D$为$x^2+y^2=R^2$圆

极坐标换元得到$\iint_D(R^2-x^2-y^2)^{\frac{3}{2}}\mathrm{d}x\mathrm{d}y=\int_0^{2\pi}\mathrm{d}\theta\int_0^R(R^2-r^2)^{\frac{3}{2}}r\mathrm{d}r=\frac{2}{5}\pi R^5$

由对称性得到原式$=3\times2\times\frac{2}{5}\pi R^5=\frac{12}{5}\pi R^5$

- **三个公式**

格林公式:P,Q连续可微$\int_{\partial D^+} P\mathrm{d}x+Q\mathrm{d}y=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y=\iint_D\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ P & Q\end{vmatrix}\mathrm{d}x\mathrm{d}y$

或$\int_{\partial D^+}(P\cos\langle\vec{n},x\rangle+Q\cos\langle\vec{n},y\rangle)\mathrm{d}s=\iint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\mathrm{d}x\mathrm{d}y$,$\vec{n}$为外法线方向

∂D^+正方向的规定:沿着曲线行进时区域在左边

高斯公式:$\iint_{\partial \mathrm{V}} P\mathrm{d}y\mathrm{d}z+Q\mathrm{d}z\mathrm{d}x+R\mathrm{d}x\mathrm{d}y=\iiint_V\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iiint_V \operatorname{div}\vec{u}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$

斯托克斯公式:$\int_{\partial S} P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z=\iint_S\begin{vmatrix}\mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x & \mathrm{d}x\mathrm{d}y\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$

∂S的方向由S的定向通过右手螺旋准则确定

- **例 1**

计算$\iint_S x^3\mathrm{d}y\mathrm{d}z+y^3\mathrm{d}z\mathrm{d}x+z^3\mathrm{d}x\mathrm{d}y$,$S$为球面$x^2+y^2+z^2=R^2$的外侧

利用高斯公式,原式$=\iiint_V 3(x^2+y^2+z^2)\mathrm{d}V$,$V$为球$x^2+y^2+z^2=R^2$

于是原式$=3\times 2\pi\int_0^R r^4\mathrm{d}r\int_0^\pi \sin\theta\,\mathrm{d}\theta=\frac{12}{5}\pi R^5$

- **例 2**

计算$\int_L(y-z)\mathrm{d}x+(z-x)\mathrm{d}y+(x-y)\mathrm{d}z$,$L$为$x^2+y^2=R^2$与$\frac{x}{a}+\frac{z}{h}=1$ 的交线

从Ox轴正向看去,椭圆是逆时针方向进行的

利用斯托克斯公式,原式$=-2\iint_S\mathrm{d}y\mathrm{d}z+\mathrm{d}z\mathrm{d}x+\mathrm{d}x\mathrm{d}y$,取$S$为该椭圆

利用投影计算$\iint_S\mathrm{d}y\mathrm{d}z=\pi\frac{Rh}{a}R=\frac{\pi R^2h}{a},\iint_S\mathrm{d}z\mathrm{d}x=0,\iint_S\mathrm{d}x\mathrm{d}y=\pi R^2$

原式$=-2\pi R^2\frac{h+a}{a}$

- **极限过渡**

$(a1)F_t$在B_t上一致收敛 $(a2)F_t$在B_x上极限存在$\Rightarrow\lim\limits_{B_x}\lim\limits_{B_t}F_t(x)=\lim\limits_{B_t}\lim\limits_{B_x}F_t(x)$

$(b1)F_t$在B_t上一致收敛 $(b2)F_t$在x_0连续$\Rightarrow F$在x_0连续

$(c1)F_t$在B_t上一致收敛 $(c2)F_t$在某区间上可积$\Rightarrow F$在该区间上也可积

$(d1)\boldsymbol{F'_t}$在B_t上一致收敛于g $(d2)F_t$至少在一点x_0收敛$\Rightarrow F_t$一致收敛于F, 且$F' = g$

- **另外两个例子**

➢ **例 *1***

求曲面围成的体积$x^2+y^2+z^2=a^2, x^2+y^2 \le a|x|$

$r^2\sin^2\theta \le ar\sin\theta\cos\varphi \Rightarrow r\sin\theta \le a\cos\varphi$

$$V = 8\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \left(\int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} \mathrm{d}\varphi \int_0^{\frac{a\cos\varphi}{\sin\theta}} r^2\sin\theta\,\mathrm{d}r + \int_0^{\frac{\pi}{2}-\theta} \mathrm{d}\varphi \int_0^{a} r^2\sin\theta\,\mathrm{d}r\right)$$

$$= 8\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \left(\int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} \frac{a^3\cos^3\varphi}{3\sin^2\theta}\mathrm{d}\varphi + \int_0^{\frac{\pi}{2}-\theta} \frac{1}{3}a^3\sin\theta\,\mathrm{d}\varphi\right)$$

$$= 8\int_0^{\frac{\pi}{2}} \left(\frac{a^3(\frac{2}{3}-\cos\theta+\frac{1}{3}\cos^3\theta)}{3\sin^2\theta} + \frac{1}{3}a^3\left(\frac{\pi}{2}-\theta\right)\sin\theta\right)\mathrm{d}\theta$$

$$= \frac{8}{3}a^3\int_0^{\frac{\pi}{2}} \left(\frac{\frac{2}{3}-\cos\theta+\frac{1}{3}\cos^3\theta}{\sin^2\theta} + \frac{\pi}{2}\sin\theta - \theta\sin\theta\right)\mathrm{d}\theta$$

$$= \frac{8}{3}a^3\left(\left(-\frac{2}{3}\cot\theta + \frac{1}{\sin\theta} - \frac{1}{3\sin\theta} - \frac{\sin\theta}{3}\right)\Big|_0^{\frac{\pi}{2}} + \frac{\pi}{2} - \left(-\theta\cos\theta\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\cos\theta\,\mathrm{d}\theta\right)\right)$$

$$= \frac{8}{3}a^3\left(\left(\frac{2(1-\cos\theta)}{3\sin\theta}\right)\Big|_0^{\frac{\pi}{2}} - \frac{1}{3} + \frac{\pi}{2} - 1\right)$$

$$= \frac{8}{3}a^3\left(\frac{2}{3} - \frac{1}{3} + \frac{\pi}{2} - 1\right)$$

$$= \frac{8}{3}a^3\left(\frac{\pi}{2} - \frac{2}{3}\right)$$

➢ **例 2**

$$I=\int_0^1 \mathrm{d}x\int_x^1 \mathrm{d}y\int_y^1 y\sqrt{1+z^4}\mathrm{d}z=\int_0^1 \mathrm{d}z\int_0^z \mathrm{d}y\int_0^z y\sqrt{1+z^4}\,\mathrm{d}x$$

$$=\int_0^1 \mathrm{d}z\int_0^z zy\sqrt{1+z^4}\mathrm{d}y$$

$$=\int_0^1 \frac{1}{2}z^3\sqrt{1+z^4}\mathrm{d}z$$

$$=\frac{1}{8}\int_1^2 \sqrt{1+z^4}\mathrm{d}(1+z^4)=\frac{1}{12}(1+z^4)^{\frac{3}{2}}\Big|_0^1=\frac{1}{12}(2\sqrt{2}-1)$$

- **反常积分极限过渡**

$(a1)f(x,y)$在B_y上一致收敛于$g(x)$

$(a2)f(x,y)$关于x反常积分一致收敛

$\Rightarrow g(x)$反常可积，$\lim\limits_{B_y}\int_a^w f(x,y)\,\mathrm{d}x=\int_a^w g(x)\mathrm{d}x$

$(b1)f(x,y)$连续

$(b2)f(x,y)$关于x反常积分一致收敛

$\Rightarrow f$可积且$f(x,y)=\int_a^w \mathrm{d}x\int_c^d f(x,y)\mathrm{d}y$

$(c1)f(x,y),f_y'(x,y)$连续

$(c2)f_y'(x,y)$关于x反常积分一致收敛

$(c3)f(x,y)$关于x反常积分至少在一点y_0收敛

$\Rightarrow f(x,y)$*关于*x*反常积分收敛于*$F(y)$*且*$F'(y)=\int_a^w f_y'(x,y)\,\mathrm{d}x$

$(d1)f(x,y)$连续

$(d2)f(x,y)$*关于每个变量的反常积分都一致收敛*

$(d3)\int_a^w \mathrm{d}x\int_c^{\widetilde{w}}|f(x,y)|\mathrm{d}y$ 或 $\int_c^{\widetilde{w}}\mathrm{d}y\int_a^w|f(x,y)|\mathrm{d}x$ 存在

$\Rightarrow \int_a^w \mathrm{d}x\int_c^{\widetilde{w}} f(x,y)\mathrm{d}y=\int_c^{\widetilde{w}}\mathrm{d}y\int_a^w|f(x,y)|\mathrm{d}x$ 反常积分可交换